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Quantum Game Theory

by

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Quantum game theory is the study of strategic behavior by agents with access to quantum technology. Broadly speaking, this technology can be employed in either of two ways: As part of a randomization device or as part of a communications protocol.

When it is used for randomization, quantum technology allows players to coordinate their strategies in certain ways. The equilibria that result are all correlated equilibria in the sense of Aumann [A], but they form a particularly interesting subclass of correlated equilibria, namely those that are both achievable and deviation-proof when players have access to certain naturally defined technologies. Not all correlated equilibria can be implemented via quantum strategies, and of those that can, not all are quantum-deviation-proof.

When players have access to private information, the theories of correlated and quantum equilibrium diverge still further, with the classical equivalence between mixed and behavioral strategies breaking down, and the appearance of new equilibria that have no classical counterparts.

The second game theoretic application of quantum technology, other than randomization, is to communication. This leads to a new set of equilibria that seem to have no natural interpretation in terms of correlated equilibria or any other classical concepts.

In Section I below, we will review the elements of game theory, with special emphasis on those aspects that are generalized or modified by quantum phenomena. In Section II, we survey games with quantum randomization and in Section III, we survey games with quantum communication.

I. Game Theory

IA. Games.

A *two-person game* consists of two sets S_1 and S_2 and two maps

$$P_1 : S_1 \times S_2 \to \mathbf{R} \qquad\qquad P_2 : S_1 \times S_2 \to \mathbf{R}$$

where **R** denotes the real numbers. The sets S_i are called *strategy sets* and the functions P_i are called *payoff functions*. Games are intended to model strategic interactions between agents who are usually called *players* (though the players are not part of this formal definition). The value $P_i(x, y)$ is called the *payoff* to Player i when Player 1 chooses strategy x and Player 2 chooses strategy y.

For simplicity, we will usually assume the sets S_i are finite.

A *solution concept* is a function that associates to each game a subset of $S_1 \times S_2$; the idea is to pick out those pairs of strategies that we believe players might select in real world situations modeled by the game. The appropriate solution concept depends on the intended application. The most studied solution concept is *Nash equilibrium.* A pair (x, y) is called a Nash equilibrium if x maximizes the function $P_1(-, y)$ and y maximizes the function $P_2(x, -)$.

IB. Mixed Strategies

To provide an accurate model of real world strategic situations, we must allow for the possibility that players might bend the rules. For example, instead of choosing a single strategy, one or both players might randomize. Starting with a game **G**, we model this possibility by constructing the *associated mixed strategy game* $\mathbf{G}^{\textbf{mixed}}$ in which the strategy space S_i is replaced with the set $\Omega(S_i)$ of probability distributions on S_i, and the payoff function P_i is replaced with the function

$$P_i^{\text{mixed}} : (\mu, \nu) \mapsto \int P_i(x, y) d\mu(x) d\nu(y)$$

Although $\mathbf{G}^{\textbf{mixed}}$ is not the same game as **G**, it is traditional to refer to a Nash equilibrium in $\mathbf{G}^{\textbf{mixed}}$ as a *mixed strategy equilibrium* in the game **G**.

An alternative but equivalent model would allow player i to choose not a probability distribution on S_i but an S_i-valued random variable from some allowable set. This is the approach we will generalize in what follows.

IC. Correlated Strategies

In the play of $\mathbf{G}^{\textbf{mixed}}$, we can imagine that Player i first selects a probability distribution, then selects a random variable (with values in S_i) that realizes that distribution, then observes a realization of that random variable, and then plays accordingly. Implicit in this description is that the random variables available to Player 1 are statistically independent of those available to Player 2.

In the real world, however, this isn't always true. In the extreme case, both agents might be able to observe the *same* random variable—such as the price of wheat as reported in the *Wall Street Journal*. We can model this extreme case by replacing $\mathbf{G}$ with a set of games $\{\mathbf{G}_\alpha\}$, one for each value α of the jointly observed random variable. We then analyze each game $\mathbf{G}_\alpha$ separately.

But in less extreme cases, we need a new concept. An *environment* for $\mathbf{G}$ is a pair $(\mathcal{X}_1, \mathcal{X}_2)$ where $\mathcal{X}_i$ is a set of S_i-valued random variables. (We do *not* assume that random variables in $\mathcal{X}_1$ are necessarily independent of those in $\mathcal{X}_2$). Given such an environment, we define a new game $\mathbf{G}(E) = \mathbf{G}(\mathcal{X}_1, \mathcal{X}_2)$ as follows:

A strategy for Player i is a random variable of the form $\sigma \circ X$ where X is an element of X_i and σ is a map from S_i to itself. (The function σ models the fact that players can map realizations of random variables to strategies any way they want to). The payoff functions are defined in the obvious way, namely:

$$P_i(\sigma \circ X, \tau \circ Y) = \int_{\mathcal{S}_1 \times \mathcal{S}_2} P_i(x, y) d\mu_{(\sigma \circ X, \tau \circ Y)}(x, y)$$

where $\mu_{(\sigma \circ X, \tau \circ Y)}$ is the joint probability distribution on $S_1 \times S_2$ induced by $(\sigma \circ X, \tau \circ Y)$

Definition IC.1. The pair of random variables (X, Y) is called a *correlated equilibrium* in $\mathbf{G}$ if it is a Nash equilibrium in the game $\mathbf{G}(\{X\}, \{Y\})$. Two correlated equilibria are *equivalent* if they induce the same probability distribution on $S_1 \times S_2$. We will frequently abuse language by treating equivalent correlated equilibria as if they were identical.

It is easy to prove the following:

Proposition IC.2. Let E be an environment and suppose that (X, Y) is a Nash equilibrium in the game $\mathbf{G}(E)$. Then (X, Y) is a correlated equilibrium in $\mathbf{G}$.

However. the converse to IC.2 does not hold:

Example IC.3. Let $S_1 = S_2 = \{\mathbf{C}, \mathbf{D}\}$. Let X, Y, and W be random variables such that

$$\text{Prob}(X = W = \mathbf{C}) = \text{Prob}(X = W = \mathbf{D}) = 1/8 \qquad \text{Prob}(X \neq W = \mathbf{C}) = \text{Prob}(X \neq W = \mathbf{D}) = 3/8$$

$$\text{Prob}(Y = W = \mathbf{C}) = \text{Prob}(Y = W = \mathbf{D}) = 1/12 \qquad \text{Prob}(Y \neq W = \mathbf{C}) = \text{Prob}(Y \neq W = \mathbf{D}) = 5/12$$

Let $E = (\{X, Y\}, \{W\})$. Let $\mathbf{G}$ be the game with the following payoffs:

		Player Two	
		C	**D**
Player One	**C**	$(0,0)$	$(2,1)$
	D	$(1,2)$	$(0,0)$

It is easy to check that both (X,W) and (Y,W) yield correlated equilibria in $\mathbf{G}$. But (X,W) is not an equilibrium in the game $\mathbf{G}(E)$, though (Y,W) is.

1D. Games with Private Information.

In real world strategic interactions, either player might know something the other doesn't. We model this situation as a *game of private information*, consisting of two strategy spaces S_i, two sets (called *information sets* $\mathcal{A}_i$, a probability distribution on $\mathcal{A}_1 \times \mathcal{A}_2$, and two payoff functions

$$P_i : \mathcal{A}_1 \times \mathcal{A}_2 \times S_1 \times S_2 \to \mathbf{R}$$

Given a game of private information, the *associated game* $\mathbf{G}^{\#}$ has strategy sets $S_i^{\#} = Hom(\mathcal{A}_i, S_i)$ and payoff functions

$$P_i^{\#}(F_1, F_2) = \int_{\mathcal{A}_1 \times \mathcal{A}_2} P_i(A_1, A_2, F_1(A_1), F_2(A_2))$$

(Here $Hom(A,S)$ denotes the set of all functions from A to S.)

If $\mathbf{G}$ is a game of private information, a *Nash equilibrium* in $\mathbf{G}$ is (by definition) a Nash equilibrium in the ordinary game $\mathbf{G}^{\#}$.

Now we want to enrich the model so players can randomize. To this end, let E be an environment in the sense of Section 1C; that is, $E = (\mathcal{X}_1, \mathcal{X}_2)$ where $\mathcal{X}_i$ is a set of S_i-valued random variables. We define the *associated environment* $E^{\#} = (\mathcal{X}_1, \mathcal{X}_2)$ by setting $\mathcal{X}_i^{\#} = Hom(A_i, \mathcal{X}_i)$ and identifying the latter set with a set of $S_i^{\#}$-valued random variables.

Thus, if $\mathbf{G}$ is a game of private information with environment E, we can first "eliminate the private information" by passing to the associated game $\mathbf{G}^{\#}$ and environment $E^{\#}$, and then "eliminate the random variables" by passing to the associated game $\mathbf{G}^{\#}(E^{\#})$ as in the discussion preceding Definition 1C.1. Now we're studying an ordinary game, where we have the ordinary notion of Nash equilibrium.

Unfortunately, *this construction does not generalize to the quantum context*. To get a construction that generalizes, we need to proceed in the opposite order, by first eliminating the random variables and then eliminating the private information:

Construction ID.1. Given a game of private information $\mathbf{G}$ with an environment E, define a new game of private information $\mathbf{G}(E)$ as follows:

The information sets $\mathcal{A}_i$ and the probability distribution on $\mathcal{A}_1 \times \mathcal{A}_2$ are as in $\mathbf{G}$. A strategy for Player i is a random variable of the form $\sigma \circ X$ where X is an element of X_i, and the payoff functions are

$$P_i(A_1, A_2, \sigma \circ X, \tau \circ Y) = \int_{S_1 \times S_2} P_i(A_1, A_2, s_1, s_2) d\mu_{\sigma \circ X, \tau \circ Y)}$$

Now applying the # construction to $\mathbf{G}(E)$ gives an ordinary game $\mathbf{G}(E)^{\#}$.

Theorem. $\mathbf{G}(E)^{\#} = \mathbf{G}^{\#}(E^{\#})$.

Remark. In spirit, and in the language of [K], $\mathbf{G}^{\#}(E^{\#})$ is like the associated game with *mixed strategies*, where player i chooses a probability distribution over maps $A_i \rightarrow S_i$, while $\mathbf{G}(E)^{\#}$ is like the associated game with *behavioral strategies*, where player i chooses, for each element of A_i, a probability distribution over S_i. In [K], these games are equivalent in an appropriate sense; here they are actually the same game. The difference is that in [K], players choose probability distributions whereas here they choose random variables. But the two approaches are fundamentally equivalent.

IE. Quantum Game Theory

The notions of mixed strategy and correlated equilibria are meant to model the real-world behavior of strategic agents who have access to randomizing technologies (such as weighted coins). *Quantum game theory* is the analogous attempt to model the behavior of strategic agents who have access to quantum technologies (such as entangled particles).

Broadly speaking, there players might use these quantum technologies in either of two ways: As randomizing devices or as communication devices. We will consider each in turn.

II. Quantum Randomization

IIA. Quantum Strategies. Just as the theory of mixed strategies models the behavior of players with access to classical randomizing devices, the theory of quantum strategies models the behavior of players with access to quantum randomizing devices. These devices provide players with access to families of observable quantities that cannot be modeled as classical random variables.

For example, let X, Y, Z and W be binary random variables. Classically we have the near triviality:

$$Prob(X \neq W) \leq Prob(X \neq Y) + Prob(Y \neq Z) + Prob(Z \neq W)$$

(Proof: Imagine X, Y, Z, W lined up in a row; in order for X to differ from W, at least one of X, Y, Z, W must differ from its neighbor.) But if X, Y, Z and W are quantum mechanical measurements, this inequality need not hold. (The most obvious paradoxes are avoided by the fact that neither X and Z, nor Y and W, are simultaneously observable.)

To model the behavior of agents who can make such measurements, we can mimic the definitions (I.C.1) and (I.C.2), replacing the sets $\mathcal{X}_i$ of random variables with sets $\mathcal{X}_i$ of quantum mechanical observables. We require that any $X \in \mathcal{X}_1$ and any $Y \in \mathcal{X}_2$ be simultaneously observable. We call such a pair $E = (\mathcal{X}_1, \mathcal{X}_2)$ a *quantum environment*. (Quantum environments will be defined more precisely in Section II.B below.) Given such a quantum environment and given a game $\mathbf{G}$, we construct a new game $\mathbf{G}(E)$ just as in the discussion preceding Definition IC.1.

If (X, Y) is a Nash equilibrium in $\mathbf{G}(E)$, we sometimes speak loosely enough to say that (X, Y) is a *quantum equilibrium* in $\mathbf{G}$, though the property of being a quantum equilibrium depends not just on $\mathbf{G}$ but on the environment E. Two quantum equilibria are called *equivalent* if they induce the same probability distribution on $S_1 \times S_2$, and, as with correlated equilibria, we will sometimes speak of equivalent quantum equilibria as if they were identical.

If (X, Y) is a quantum equilibrium then (by the definition of quantum environment), X and Y are simultaneously observable and hence can be treated as classical random variables. With this identification, it is easy to show that any quantum equilibrium is a correlated equilibrium. (This generalizes Proposition IC.2.) However, just as in Example IC.3, the converse need not hold. A pair (X, Y) that is an equilibrium in one quantum environment need not be an equilibrium in another.

IIB. The Quantum Environment

Consider a game in which the strategy sets are $S_1 = S_2 = \{\mathbf{C}, \mathbf{D}\}$. Players can implement (ordinary classical) mixed strategies by flipping (weighted) pennies, mapping the outcome "heads" to the strategy **C** and the outcome "tails" to the strategy **D**.

While a classical penny occupies either the state **H**(heads up) or **T** (tails up), a quantum penny can occupy any state of the form $\psi = \alpha\mathbf{H}+\beta\mathbf{T}$, where α and β are complex scalars, not both zero. A heads/tails measurement of such a penny yields the outcome either **H** or **T** with probabilities proportional to $|\alpha|^2$ and $|\beta|^2$. Physical actions such as rotating the penny induce unitary transformations of the state space, so that the state ψ is replaced by $U\psi$ where U is some unitary operator on the complex vector space $\mathbf{C}^2$.

(Of course literal macroscopic pennies do not behave this way, but spin-1/2 particles such as electrons do, with "heads" and "tails" replaced by "spin up" and "spin down".)

A single quantum penny is no more or less useful than a classical randomizing device. If you want to play heads with probability p, you can first apply a unitary transformation that converts the state to some $\gamma\mathbf{H} + \delta\mathbf{T}$ with $|\gamma|^2/(|\gamma|^2 + |\delta^2|) = p$, then measure the heads/tails state of the penny and play accordingly.

However, two players equipped with quantum pennies have something more than a classical randomizing device. A pair of quantum pennies occupies a state of the form

$$\alpha\mathbf{H}\otimes\mathbf{H} + \beta\mathbf{H}\otimes\mathbf{T} + \gamma\mathbf{T}\otimes\mathbf{H} + \delta\mathbf{T}\otimes\mathbf{T}$$

where $\alpha, \beta, \gamma, \delta$ are complex scalars, not all zero. A physical manipulation of the first penny transforms the first factor unitarily and a physical manipulation of the second penny transforms the second factor unitarily. Subsequent measurements yield the outcomes (heads,heads), (heads,tails) and so forth with probabilities proportional to $|\alpha|^2$, $|\beta|^2$ and so forth. This allows the players to achieve joint probability distributions that cannot be achieved via the observations of independent random variables.

Example II.B.1. Suppose that two pennies begin in the *maximally entangled state* $\mathbf{H}\otimes\mathbf{H}+\mathbf{T}\otimes\mathbf{T}$. Players One and Two apply the transformations U and V to the first and second pennies where

$$U = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \qquad V = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

This converts the state from $\mathbf{H}\otimes\mathbf{H}+\mathbf{T}\otimes\mathbf{T}$ to

$$U\mathbf{H}\otimes V\mathbf{H} + U\mathbf{T}\otimes V\mathbf{T} = \cos(\theta-\phi)\mathbf{H}\otimes\mathbf{H} + \sin(\theta-\phi)\mathbf{H}\otimes\mathbf{T} - \sin(\theta-\phi)\mathbf{T}\otimes\mathbf{H} + \cos(\theta-\phi)\mathbf{T}\otimes\mathbf{T}$$

If players map the outcomes **H** and **T** to the strategies **C** and **D**, then the resulting probability distribution over strategy pairs is

$$\mathrm{Prob}(\mathbf{C},\mathbf{C}) = \mathrm{Prob}(\mathbf{D},\mathbf{D}) = \cos^2(\theta-\phi)/2 \qquad \mathrm{Prob}(\mathbf{C},\mathbf{D}) = \mathrm{Prob}(\mathbf{D},\mathbf{C}) = \sin^2(\theta-\phi)/2$$

We can now make precise the notion of *quantum environment*; a quantum enviroment is a triple $(\xi, \mathcal{X}_1, \mathcal{X}_2)$ where

a) ξ is a non-zero vector in a complex vector space $\mathbf{C}^{n_1} \otimes \mathbf{C}^{n_2}$ (with n_1 and n_2 assumed finite here, though this could all be generalized)

b) $\mathcal{X}_i$ is a set of unitary operators on $\mathbf{C}^{n_i}$.

The unitary operators fill the same role as the sets of random variables in Section I; they are the things that players can observe, and on whose realizations they can condition their strategies.

IIC. Quantum Equilibrium

Consider again the game **G** from Example IC.3:

		Player Two **C**	**D**
Player One	**C**	(0,0)	(2,1)
	D	(1,2)	(0,0)

Let E be the following quantum environment: $\xi = \mathbf{H}\otimes\mathbf{H} + \mathbf{T}\otimes\mathbf{T}$ where $\{\mathbf{H},\mathbf{T}\}$ is a basis for the complex vector space $\mathbf{C}^2$; $\mathcal{X}_1 = \mathcal{X}_2$ is the set of all operators that take the form

$$M(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

when expressed in terms of the basis $\{\mathbf{H},\mathbf{T}\}$.

Physically, this means that each player has access to one member of a pair of maximally entangled pennies, and can rotate that penny through any angle before measuring it's heads/tails orientation.

Taking Player Two's angle ϕ as given, Player One clearly optimizes by choosing θ so that $\sin(\theta-\phi) = 1$, and symmetrically for Player One. Thus in equilibium, the outcomes (2,1) and (1,2) are each realized with probability 1/2.

As we've noted earlier, this is of course a correlated equilibrium, but it is more than that. For example, the correlated equilibria (X, W) and (Y, W) of Example IC.3 are not sustainable as quantum equilibria in this environment.

IID. Quantum Games of Private Information

Let $\mathbf{G}$ be a game of private information as defined in Section ID, and let E be a quantum environment.

We would like to model the play of $\mathbf{G}$ in the environment E as the play of an ordinary game. A natural attempt is to replace $\mathbf{G}$ with the game $\mathbf{G}^{\#}$ defined in Section ID; recall that a strategy in $\mathbf{G}^{\#}$ is a map from the information set $\mathcal{A}_i$ to the strategy set S_i. However, in the quantum case there is no natural way to define an environment $E^{\#}$ for this game. More precisely, it is shown in [DL] that the natural definition of $E^{\#}$ makes sense when and only when the measurements in $\mathcal{X}_1 \cup \mathcal{X}_2$ have a single joint probability distribution, so that they can be thought of as classical random variables. In other words, the existence of an environment $E^{\#}$ is equivalent to the absence of any specifically quantum phenomena.

Therefore we must generalize not the construction $\mathbf{G}^{\#}(E^{\#})$ from Section ID, but the classically equivalent construction $\mathbf{G}(E)^{\#}$. In the language of [K] (and of the remark at the end of Section ID), quantum game theory allows players to choose behavioral strategies without equivalent mixed strategies.

Example IID.1. (This example is adapted from [CHTW].) Let $\mathbf{G}$ be the following game of private information:

The information sets are $\mathcal{A}_1 = \mathcal{A}_2 = \{\text{red,green}\}$. The probability distribution on $\mathcal{A}_1 \times \mathcal{A}_2$ is uniform. The payoff functions are specified as follows:

IF BOTH PLAYERS OBSERVE RED

Player One \ Player Two	**C**	**D**
C	$(1,1)$	$(0,0)$
D	$(0,0)$	$(1,1)$

IF EITHER PLAYER OBSERVES GREEN

Player One \ Player Two	**C**	**D**
C	$(0,0)$	$(1,1)$
D	$(1,1)$	$(0,0)$

The environment E is as in IIC.

In the game $\mathbf{G}(E)^{\#}$, players choose maps $\mathcal{A}_i \to \mathcal{X}_i$.

To find equilibria in this game, suppose that Players One has chosen to map "red" and "green" to $M(\theta_{red})$ and $M(\theta_{green})$, while Player Two has chosen $M(\phi_{red})$ and M(ϕ_{green}) (where the M matrices are as defined in Section IC.) Then we can compute Player One's payoffs as functions on $\mathcal{A}_1 \times \mathcal{A}_2$:

$$EP_1(\text{red}, \text{red}) = \cos^2(\theta_{red} - \phi_{red}) \tag{IID.1}$$

$$EP_1(\text{red}, \text{green}) = \sin^2(\theta_{red} - \phi_{green}) \tag{IID.2}$$

$$EP_1(\text{green}, \text{red}) = \sin^2(\theta_{green} - \phi_{red}) \tag{IID.3}$$

$$EP_1(\text{green}, \text{green}) = \sin^2(\theta_{green} - \phi_{green}) \tag{IID.4}$$

Because the probability distribution on $\mathcal{A}_1 \times \mathcal{A}_2$ is uniform, Player One seeks to maximize the sum of these four expressions, taking ϕ_{red} and ϕ_{green} as given. At the same time (due to the symmetry of the problem) Player Two seeks to maximize the identical sum, taking θ_{red} and θ_{green} as given.

Given this, we can compute that there are two types of equilibria:

a) Equilibria in which exactly three of the four expressions (IID.1)-(IID.4) are equal to 1 and the fourth is equal to 0. All four possibilities occur with $\phi_r, \phi_g, \theta_r, \theta_g \in \{0, \frac{\pi}{2}, \pi\}$. In these equilibria each player receives a payoff of $3/4 = .75$.

b) Equilibria equivalent to $\{\phi_{red} = 0, \phi_{green} = 3\pi/4, \theta_{red} = \pi/8, \theta_{green} = 3\pi/8\}$. In these equilibria each player receives a payoff of $(1/2 + \sqrt{2}/4) \approx .85$.

The best equilibrium that can be reached in any classical environment is equivalent to an equilibrium of type a). This is so even when the classical environment includes correlated random variables.

For ordinary games, we observed that every quantum equilibrium is also a correlated equilibrium. For games of private information, this example demonstrates that no analogous statement is true.

III. Quantum Communication

IIIA. Communication

In any real world implementation of a game, players must somehow communicate their strategies before they can receive payoffs. This follows from the fact that the payoff functions take both players' strategies as arguments; therefore information about both strategies must somehow be present at the same place and time.

Ordinarily, the communication process is left unmodeled. But in this section, we need an explicit model so that we can explore the effects of allowing quantum communication technology. To that end we postulate a referee who communicates with the players by handing them markers (say pennies) which the players can transform from one state to another (say by flipping them over or leaving them unflipped) to indicated their strategy choices; the markers are eventually returned to the referee who examines them and computes payoffs.

Of course not all real world games have literal referees; sometimes the players are firms, their strategies are prices, and the prices are communicated not to a referee but to a marketplace, where payoffs are "computed" via market processes. In the models to follow, the referee can be understood as a metaphor for such processes.

IIIB. A Quantum Move

Consider the game with the following payoff matrix (for now, view the labels **NN**, **NF**, etc., as arbitrary labels for strategies):

		Player Two	
		N	**F**
Player One	**NN**	$(1,0)$	$(0,1)$
	NF	$(0,1)$	$(1,0)$
	FN	$(0,1)$	$(1,0)$
	FF	$(1,0)$	$(0,1)$

As noted above, classical game theory does not ask how players communicate with the referee. If we want to embellish our model with an explicit communication protocol, there are several essentially equivalent ways to do it.

IIIB.1. The Simplest Protocol: The referee passes two pennies to Player One, who flips both to indicate a play of **FF**, flips only the first to indicate a play of **FN**, and so forth, and one penny to Player Two, who flips (**F**) or does not (**N**). The pennies are returned to the referee, who examines their states and makes payoffs accordingly.

IIIB.2. An Alternative Protocol: A single penny in state **H** is passed to Player One, who either flips or doesn't; the penny is then passed to Player Two, who either flips or doesn't; the penny is then returned to the Player One, who either flips or doesn't; the penny is then returned to the referee who makes payoffs of (1,0) if the final state is **H** or (0,1) if the final state is **T**. (The players are blindfolded and cannot observe each others' plays.)

We can set things up so that flipping and not-flipping correspond to the applications of the unitary matrices

$$\mathbf{F} = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now suppose that Player One manages to cheat by employing arbitrary unitary operations. Under the simplest protocol IIIB.1, this is equivalent to playing a mixed strategy and gives Player One no advantage. But under the alternative protocol, Player One can guarantee himself a win by choosing the unitary matrix

$$U = \frac{1}{2}\begin{pmatrix} 1+i & \sqrt{2} \\ -\sqrt{2} & 1-i \end{pmatrix}$$

on his first turn and U^{-1} on his second. This guarantees him a win because $U^{-1} \circ F \circ U$ and $U^{-1} \circ N \circ U = N$ are both diagonal matrices, so that both fix the state **H**. (Recall that states are unchanged by scalar multiplication.) In other words, the final state is now **H** regardless of Player Two's strategy.

Note that it might be quite impossible to prohibit Player One from employing the matrix U, for exactly the same reason that it is usually quite impossible to prohibit players for adopting mixed strategies: When the referee makes his final measurement, the only information revealed is the final state of the penny, not the process by which it achieved that state.

This example, due to David Meyer ([M]), illustrates two points: First, quantum communication can matter. Second, whether or not quantum communication matters depends not just on the game **G**; it depends on the specified communications protocol.

IIIC. The Eisert-Wilkens-Lewenstein Protocol

The Eisert-Wilkens-Lewenstein protocol ([EW],[EWL]) captures the potential effects of quantum communication in a quite general context and is therefore the most studied protocol in games of quantum communication.

We start with a game **G** in which the strategy sets are $S_1 = S_2 = \{\mathbf{C}, \mathbf{D}\}$. (Everything can be generalized to larger strategy sets, but we will stick to this simplest case.) The referee prepares a pair of pennies in the state $\mathbf{H} \otimes \mathbf{H}+\mathbf{T} \otimes \mathbf{T}$, and passes one penny to each player, with Player One receiving the "left-hand" penny.

Players are instructed to play operate with one of the unitary matrices **F** and **N** of Section IIIB, with **F** indicating a desire to play **D** and **N** a desire to play **C**. Players actually operate with the unitary matrices of their choices and then return the pennies to the referee, who makes a measurement that distinguishes among the four states that could result if the players followed instructions.

We can of course model this situation by saying that the original game **G** has been replaced by the *associated quantum game* $\mathbf{G}^Q$ (not to be confused with the associated quantum games that arise in the very different context of Section II), with strategy sets equal to the unitary group U_2 (or, equivalently— because states are unchanged by scalar multiplication— the special unitary group SU_2) and payoff functions have the obvious definition. It is observed in [L] that $\mathbf{G}^Q$ is equivalent to the following game:

- The strategy sets S_i^Q are both equal to the group of unit quaternions (which is isomorphic to the special unitary group SU_2)
- The payoff functions are defined as follows:

$$P_i^Q(\mathbf{p},\mathbf{q}) = A^2P_i(\mathbf{C},\mathbf{C}) + B^2P_i(\mathbf{C},\mathbf{D}) + C^2P_i(\mathbf{D},\mathbf{C}) + D^2P_i(\mathbf{D},\mathbf{D})$$

  where P_i is the payoff function in **G** and where

$$\mathbf{pq} = A + Bi + Cj + Dk$$

The group structure on the strategy sets guarantees that (except in the uninteresting case where both payoff functions are maximized at the same arguments) there can be no pure-strategy equilibria in this game, because Player One, taking Player Two's strategy **q** as given, can always choose **p** to maximize his own payoff, in which case Player Two cannot be maximizing. So the game $\mathbf{G}^Q$ is quite uninteresting unless we allow mixed strategies.

A mixed strategy in $\mathbf{G}^Q$ is a probability measure on the strategy space $S_1^Q = SU_2 = \mathbf{S}^3$ where $\mathbf{S}^3$ is the three-sphere. Thus the strategy spaces in $(\mathbf{G}^Q)^{\mathbf{mixed}}$ are quite large. However, it is shown in [L] that in equilibrium, both players can be assumed to adopt strategies supported on at most four points, and that each set of four points must lie in one of a small number of highly restrictive geometric configurations. This considerably eases the problem of searching for equilibria.

Example IIIC.1. The Prisoner's Dilemma. Consider the "Prisoner's Dilemma" game

		Player Two	
		C	**D**
Player One	**C**	$(3,3)$	$(0,5)$
	D	$(5,0)$	$(1,1)$

There is only one Nash equilibrium, only one (classical) mixed strategy equilibrium, and only one (classical) correlated equilibrium, namely $(\mathbf{D},\mathbf{D})$ in every case. But it is an easy exercise to check that there are multiple mixed-strategy equilibria in the Eisert-Wilkens-Lewenstein game $\mathbf{G}^Q$.

First example: Each player chooses any four orthogonal quaternions and plays each of the four with equal probability. This induces the probability distribution in which each of the four payoffs is equiprobable and each player earns an expected payoff of 9/4.

Second example: Player One plays the quaternions 1 and i with equal probability and Player Two plays the quaternions j and k with equal probability. This induces the probability distribution where the payoffs $(0,5)$ and $(5,0)$ each occur with probability 1/2 so that each player earns an expected payoff of 5/2.

The techniques of [L] reveal that, up to a suitable notion of equivalence, these are the only mixed strategy equilibria in $\mathbf{G}^Q$.

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