if $y \in C$, then the component of x_k in the expansion of y with respect to B is bounded in the interval $[0, k]$. This makes it impossible for C to contain a ray since any ray in the subspace generated by B has the form

$$\left\{ \sum_1^N (a_k + \lambda b_k) x_k : \lambda \geq 0 \right\},$$

where N is chosen sufficiently large and at least one of the b_k's is nonzero.

The necessity follows from the two lemmas.

THE JIMMY'S BOOK

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In 1974, I had the great good fortune to come to the University of Chicago as a graduate student. Our class soon distinguished itself by a prodigious capacity for bridge and by the offbeat problems that we invented to challenge each other. To the best of my knowledge, none of these problems was ever solved, although a few enjoyed the status of mass obsessions for several years.

None of us thought to record any of the problems or our partial solutions at the time; if we had, they would have filled a small book. By analogy with [**1**], and in honor of a venerable Hyde Park institution, it would have to have been called "The Jimmy's Book". My purpose here is to record a few of the most memorable entries from that book that was never written.

1. Zero-One Matrices. This problem was first posed by Michael Stob, who had heard of its appearance on an exam for high school students! Consider the set of $n \times n$ matrices all of whose entries are either zero or one. How many of these 2^{n^2} matrices have determinant one?

It may be that more time was invested in this problem than in all the others put together. We sat up many long nights, sometimes alone and sometimes in small groups, trying one attack after another. No significant progress was ever made.

One can ask the apparently easier question of how many of these matrices have determinant zero. I once (the night before Yitz Herstein's Algebra exam, in fact) computed the 5×5 case by hand, using a method that filled most of a notebook and didn't seem to generalize. Yitz was anxious to discuss all of this while the exam was in progress, but discussion was finally postponed till later in the afternoon.

A number of related questions were discussed as well. Mike Stob raised the question: What is the largest possible determinant for an $n \times n$ matrix of zeros and ones? At one point, he used the computer at Argonne National Lab to generate 10,000 random 10×10 zero-one matrices and calculate their determinants. To the best of my recollection, the largest determinant that occurred was 24, and it occurred exactly once.

2 The Topology of the Croke Space. The Croke Space C is defined by starting with the Riemann Sphere and identifying points that are roots of the same irreducible polynomial over the rationals. What can be said about the topology of this space? In particular, compute its homology and homotopy groups. (Peter Johnstone observed that C cannot be Hausdorff, since there are irrational square roots of rational numbers arbitrarily close to π, and these are identified with their negatives while π is not.) This problem was posed by Chris Croke. I think it was Kathy Edwards who named the space in question for its inventor.

3. The Locally / Not Problem. This problem came from Peter Johnstone, who was then an instructor at Chicago. I seem to recall that he attributed it to someone else, but I can't remember who. Let P be a property of topological spaces, and say that a space is "locally P" if each point has a base of neighborhoods all having property P. Now choose a particular property, such as connectedness. Consider successively the class of all spaces that are

(a) connected
(b) locally connected
(c) not locally connected
(d) locally not locally connected
(e) not locally not locally connected

et cetera.

How many of these classes are distinct?

One can also consider the class of all spaces that are

(a′) connected
(b′) not connected
(c′) locally not connected
(d′) not locally not connected

and so forth, and one can ask the same question. Now repeat with connected replaced by compact, or Hausdorff, or noetherian, or . . . .

Just visualizing some of these classes is already a somewhat ambitious project. As Peter observed at the time, it is not at first obvious what it means for a space to be locally not locally not connected.

4. The Exam Question. Let m, n and k be positive integers. What is the cardinality of the smallest set having m subsets of cardinality n, no two of which have more than k elements in common?

This problem was also posed by Mike Stob, who had a practical application for it: he wanted to give each of 15 students in his Calculus class a personalized take-home exam with 20 problems, and to minimize cheating he wanted no two exams to have more than 5 problems in common. How many problems would he have to make up?

Lauren Feinstone, who was then a graduate student in economics, devised an algorithm which appeared to yield a good upper bound. Unfortunately, computing this bound for the particular values of m, n and k that Mike was interested in turned out to be impracticably complicated.

5. The Word Problem. This was the granddaddy of them all. It was inherited from the more advanced graduate students, in whose lives it had played the same role that zero-one matrices played in ours.

Consider the free group on twenty-six letters $A, B, C, \ldots, Z$. Mod out by the relation that defines two words to be equivalent if (a) one is a permutation of the other and (b) each appears as a legitimate English word in the dictionary. (Obviously this definition may not be invariant under change of dictionary. I hereby arbitrarily and immutably standardize it by naming [**2**] as the unconditional arbiter.) Identify the center of this group.

The fourth floor of Eckhart Hall once housed an enormous chart with 26 rows and columns. In the (i, j)-spot was recorded a proof that letter i commutes with letter j. Some spots were blank, but most were filled in. Several individual letters had been shown to lie in the center. On the other hand, it is very unlikely that the group is actually abelian: it is difficult to imagine a proof that Q commutes with X, for example.

I don't know what became of that chart. If it's been lost, a lot of effort and a lot of history are lost with it.

Other problems that we worked on were less outré and must have been considered by many people besides ourselves. We used to try to count both the number of distinct topologies and the number of distinct homeomorphism types of topologies on a set of n points. We once killed an afternoon constructing a non-zero polynomial over the quaternions all of whose values are zero (I think we had some help from Kaplansky), and asking exactly which skew fields it's possible to do this for. We also worked on the Riemann Hypothesis and the Poincaré Conjecture, without notable result.

(I fondly remember when Chris Croke proved the Poincaré Conjecture, modulo two facts that seemed plausible. He showed me the proof, we couldn't find anything wrong with it, and we went to ask Dick Lashof about the status of the two "facts". His response: "If those things were true, you could prove the Poincaré Conjecture".)

It's also true that each problem had an associated meta-problem: concoct a plausible scenario in which a mathematician would need to know the answer to this problem. It was generally our conviction that none of the meta-problems was solvable, but I remember a conversation in which Jon Alperin attacked them all with vigor. As I kept challenging him with more and more contrived and outlandish problems, he kept managing to argue that this, too, was a problem whose solution might someday be valuable. I finally stumped him with the Word Problem. He had to admit that it was inconceivable that this one could have any significance for anything. But I think he liked the problem, anyway.

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MAXIMUM PRODUCTS AND $\lim\left(1+\frac{1}{n}\right)^n = e$

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In a recent edition of this Monthly [1] there appears an article by C. W. Barnes entitled "Euler's constant and e" in which he deduces the existence of Euler's constant from the fact that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ or, to be more precise, from the inequality

$$e\bigg/\left(1+\frac{1}{n}\right) < \left(1+\frac{1}{n}\right)^n < e. \tag{1}$$

He also gives a simple and 'natural' proof of (1), asserting that there is a dearth of such proofs in the textbook literature. We give here a very simple proof of (1) that has a motivation that we think is natural and interesting.

We first give the bare bones of the proof and later fill it out with the flesh of motivation. As in Professor Barnes' proof, the only properties we use of e and the logarithmic function are $\ln e = 1$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$, and $\ln x^n = n \ln x$ for $n = 1, 2, \ldots$.

Let $f_n(t) = t\ln(ne/t)$, $t > 0$. Then $f_n'(t) = \ln(ne/t) - 1$. So f_n has its maximum value at $t = n$. Thus

$$f_n(n+1) < f_n(n), \quad \text{i.e.,} \quad \left(\frac{ne}{n+1}\right)^{n+1} < \left(\frac{ne}{n}\right)^n, \quad \text{i.e.,} \quad e\bigg/\left(1+\frac{1}{n}\right) < \left(1+\frac{1}{n}\right)^n.$$

Also

$$f_{n+1}(n) < f_{n+1}(n+1), \quad \text{i.e.,} \quad \left(\frac{(n+1)e}{n}\right)^n < \left(\frac{(n+1)e}{n+1}\right)^{n+1}, \quad \text{i.e.,} \quad \left(1+\frac{1}{n}\right)^n < e.$$

The motivation for this proof is the following problem: given a number $X > 1$, how can we